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The geometric sense of the Sasaki connection

Alexey V Shchepetilov

Department of Physics, Moscow State University, 119992 Moscow, Russia

E-mail: alexey@quant.phys.msu.su

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Abstract

For the Riemannian manifold M^n two special connections are constructed on the sum of the tangent bundle TM^n and the trivial one-dimensional bundle. These connections are flat if and only if the space M^n has a constant sectional curvature ± 1. The geometric explanation of this property is given. This construction gives a coordinate-free many-dimensional generalization of the Sasaki connection (Sasaki R 1979 Soliton equations and pseudospherical surfaces *Nucl. Phys.* B **154** 343–57). It is shown that these connections have a close relation to the imbedding of M^n into Euclidean or pseudo-Euclidean $(n + 1)$-dimension spaces.

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1. Introduction

In 1979, Sasaki [1] proposed the formula for some local connection on a two-dimensional real Riemannian manifold M^2, which has played a big role in the theory of nonlinear integrable partial differential equations. The construction of this connection is as follows[1].

Let g be a Riemannian metric on M^2, let ∇ be the corresponding Levi-Civita connection on TM^2, let $\{e_1, e_2\}$ be a moving orthonormal frame on some open domain $U \subset M^2$ and let $\{\omega^1, \omega^2\}$ be a corresponding moving coframe. The relations $\nabla(e_i) = \omega_i^j \otimes e_j$ define the connection 1-form matrix ω_i^j with respect to the frame $\{e_1, e_2\}$. The orthonormality of this frame implies that $\omega_1^1 = \omega_2^2 = 0, \omega_1^2 = -\omega_2^1$. The Levi-Civita connection is torsion-free that implies the following structural equations:

$$d\omega^1 = \omega^2 \wedge \omega_2^1 \qquad d\omega^2 = \omega^1 \wedge \omega_1^2. \tag{1}$$

The Gaussian curvature K of the space M^2 is defined by

$$d\omega_2^1 = K\omega^1 \wedge \omega^2. \tag{2}$$

[1] The following description of the Sasaki construction is slightly different from the original one for the better agreement with the sequel. Particularly, we choose the sign in the null curvature condition in a more geometric way.

Sasaki proposed the consideration of a matrix 1-form on M^2

$$A = \frac{1}{2}\begin{pmatrix} \omega^2 & -\omega^1 + \omega_2^1 \\ -\omega^1 - \omega_2^1 & -\omega^2 \end{pmatrix} \tag{3}$$

as a new connection form for some (non-specified) bundle over M^2. The key property of the matrix 1-form A is that it satisfies the null curvature condition $\Omega \equiv dA + A \wedge A \equiv 0$ iff $K \equiv -1$ on U.

In some preceding (e.g. [2]) and many consequent papers (some of the most recent are [3–7]), different matrix 1-forms have been discussed, depending on a function u (or some functions) of some independent variables, such that the null curvature condition for this form is equivalent to one of the well-known nonlinear partial differential equations (KdV, mKdV, sine-Gordon, sinh-Gordon, nonlinear Schrödinger, Burgers) possessing many conservation laws and reach symmetry groups. Sasaki was the first to connect the matrix 1-form A with the surfaces of constant negative curvature. In their paper, Chern and Tenenblat [8] defined a class of differential equations $\mathcal{F}[u] = 0$, which could be obtained as a null curvature condition for the form (3), depending on the function u.

However, it seems that the geometric meaning of the connection $\tilde{\nabla}^h$, corresponding to the matrix 1-form A, remained unclear. Firstly, the definition (3) is valid only for a local trivialization of a potential unknown bundle, because it might be that there is no global moving frame on M^2, for example for $M^2 = \mathbb{S}^2$. Secondly, according to equation (3) the matrix 1-form A is a $\mathfrak{sl}(2, \mathbb{R})$-valued 1-form. This seems slightly strange, because it is defined for an arbitrary metric g, and the corresponding group $\mathrm{SL}(2, \mathbb{R})$ is the isometry group only for M^2 equal to the hyperbolic plane $\mathbb{H}^2$. Thirdly, the forms ω^1, ω^2 and ω_2^1 play a similar role in equation (3), but their geometric sense is quite different. The Levi-Civita connection 1-form for the tangent bundle TM^2 with respect to the moving frame $\{e_1, e_2\}$ is

$$\begin{pmatrix} 0 & \omega_2^1 \\ -\omega_2^1 & 0 \end{pmatrix}. \tag{4}$$

It is contained in A with the strange factor $\frac{1}{2}$, violating the geometric sense of this summand due to the nonlinear relation between a connection 1-form A and a curvature 2-form Ω. Finally, it seems to be unclear how to generalize this construction for higher dimensions. Below we answer all these questions.

Note the difference of the Sasaki connection from the Sasaki geometry introduced in [9].

2. Reformulation of the Sasaki construction

We denote

$$\sigma_1 = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \qquad \sigma_2 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \sigma_3 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

the base in the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$. The commutative relations for this are

$$[\sigma_1, \sigma_2] = \sigma_3 \qquad [\sigma_2, \sigma_3] = -\sigma_1 \qquad [\sigma_3, \sigma_1] = -\sigma_2. \tag{5}$$

Then the connection matrix 1-form (3) can be expressed as

$$A = \sigma_1\omega^1 + \sigma_2\omega^2 + \sigma_3\omega_2^1 \tag{6}$$

and the corresponding curvature form will be

$$\begin{aligned}\Omega = A + \tfrac{1}{2}[A, A] &= \sigma_1 d\omega^1 + \sigma_2 d\omega^2 + \sigma_3 d\omega_2^1 + [\sigma_1, \sigma_2]\omega^1 \wedge \omega^2 \\ &\quad + [\sigma_1, \sigma_3]\omega^1 \wedge \omega_2^1 + [\sigma_2, \sigma_3]\omega^2 \wedge \omega_2^1.\end{aligned}$$

Here we used a standard notation $[B, C] = \sum_{i,j}[B_i, C_j]\omega^i \wedge \omega^j$, where $B = \sum_i B_i\omega^i$, $C = \sum_i C_i\omega^i$; B_i, C_i are coefficients in Lie algebra, $\mathfrak{g}$ and ω^i are scalar differential 1-forms. When $\mathfrak{g}$ is a matrix algebra it is obvious that $B \wedge C = \frac{1}{2}[B, C]$. Hence the condition $\Omega = 0$ depends only on relations (1), (2) and commutative relations in the algebra $\mathfrak{sl}(2, \mathbb{R})$.

It is well known that Lie algebras $\mathfrak{sl}(2, \mathbb{R})$, $\mathfrak{so}(2, 1)$ and $\mathfrak{su}(1, 1)$ are isomorphic, so we can change elements $\sigma_1, \sigma_2, \sigma_3$ in equation (6) by the equivalent elements from $\mathfrak{so}(2, 1)$

$$\bar{\sigma}_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad \bar{\sigma}_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad \bar{\sigma}_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

with the same commutative relations (5). After this substitution, the 1-form A becomes

$$A = \bar{\sigma}_1\omega^1 + \bar{\sigma}_2\omega^2 + \bar{\sigma}_3\omega_2^1 = \begin{pmatrix} 0 & \omega_2^1 & \omega^1 \\ -\omega_2^1 & 0 & \omega^2 \\ \omega^1 & \omega^2 & 0 \end{pmatrix} \tag{7}$$

where expression

$$\bar{\sigma}_3\omega_2^1 = \begin{pmatrix} 0 & \omega_2^1 & 0 \\ -\omega_2^1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

contains the Levi-Civita connection form

$$\begin{pmatrix} 0 & \omega_2^1 \\ -\omega_2^1 & 0 \end{pmatrix} \tag{8}$$

as a direct summand. Due to this last fact, it is now possible to give a geometric interpretation of the Sasaki connection.

Let $E = \mathbb{R} \times M^2$ be a trivial one-dimensional vector bundle over M^2 and let $F = TM^2 \oplus E$ be a direct sum of two bundles over the same base. We define an indefinite metric $\tilde{g}_h$ on each fibre $T_xM^2 \oplus \mathbb{R}$ of F as a direct sum of the metric g and the metric $(x, y) = -xy$, $x, y \in \mathbb{R}$. Let e be a unit vector in $\mathbb{R}$. Thus, $\{e_1, e_2, e\}$ is a moving frame in F and the connection 1-form (7) defines a covariant derivation $\tilde{\nabla}^h$

$$\tilde{\nabla}^h e_1 = -\omega_2^1 \otimes e_2 + \omega^1 \otimes e \qquad \tilde{\nabla}^h e_2 = \omega_2^1 \otimes e_1 + \omega^2 \otimes e \qquad \tilde{\nabla}^h e = \omega^1 \otimes e_1 + \omega^2 \otimes e_2 \tag{9}$$

which conserves the metric $\tilde{g}_h$. It is easily seen that when M^2 is the hyperbolic plane $\mathbb{H}^2$, imbedded in the standard way as a one sheet of a two-sheeted hyperboloid into the pseudo-Euclidean space $\mathbb{E}^{2,1}$, then F is simply the trivial bundle $\mathbb{E}^{2,1} \times \mathbb{H}^2$. In this case E is the normal bundle over hyperboloid $\mathbb{H}^2 \subset \mathbb{E}^{2,1}$. This explains the null-curvature for $\tilde{\nabla}^h$ when $M^2 = \mathbb{H}^2$, because $\tilde{\nabla}^h$ on $\mathbb{E}^{2,1} \times \mathbb{H}^2$ is the restriction of the flat Levi-Civita connection on $T\mathbb{E}^{2,1} = \mathbb{E}^{2,1} \times \mathbb{E}^{2,1}$.

To ensure that the connection $\tilde{\nabla}^h$ is well defined on the whole bundle F for the general metric g on M^2, we can rewrite $\tilde{\nabla}^h$ as follows. Let $\xi + fe = \xi^1 e_1 + \xi^2 e_2 + fe$ be a direct expansion of an arbitrary section of F over U, where f is a smooth function on M^2 and ξ is a section of TM^2. Then due to (9) we obtain

$$\begin{aligned} \tilde{\nabla}^h_X(\xi + fe) &= X(\xi^1)e_1 + X(\xi^2)e_2 - \xi^1\omega_2^1(X)e_2 + \xi^2\omega_2^1(X)e_1 \\ &\quad + f(\omega^1(X)e_1 + \omega^2(X)e_2) + (X(f) + \xi^1\omega^1(X) + \xi^2\omega^2(X))e \\ &= \nabla_X\xi + fX + (X(f) + g(X, \xi))\, e \end{aligned} \tag{10}$$

where X is a vector field on M^2. It is obvious that this formula gives the definition for $\tilde{\nabla}^h$ on the whole bundle F.

It is possible to change the connection on the bundle F in such a way that it will be flat iff g is the metric of constant positive curvature $K = 1$. To do this, we should write the connection 1-form A as

$$A = \begin{pmatrix} 0 & \omega_2^1 & \omega^1 \\ -\omega_2^1 & 0 & \omega^2 \\ -\omega^1 & -\omega^2 & 0 \end{pmatrix}.$$

The corresponding derivation will be

$$\tilde{\nabla}_X^s(\xi + fe) = \nabla_X\xi + fX + (X(f) - g(X,\xi))\, e. \tag{11}$$

We see that now A is a $\mathfrak{so}(3)$ valued differential form and the derivation $\tilde{\nabla}^s$ conserves the positively defined metric $\tilde{g}_s$ on fibres, which is the direct sum of the metric g on TM^2 and the metric $(x, y) = xy$, $x, y \in \mathbb{R}$ on $\mathbb{R}$.

3. Generalization on higher dimensions

The formulae (10) and (11) make it possible to immediately generalize this construction on higher dimensions. Now M^n becomes an n-dimensional Riemannian manifold and F is the bundle $TM^n \oplus E$, where $E = M^n \times \mathbb{R}$. We define the connections $\tilde{\nabla}^s$ by equation (11) and connection $\tilde{\nabla}^h$ by the right-hand side of equation (10), where e again is the unit element of the fibre $\mathbb{R}$ of F. The definitions for metrics $\tilde{g}_h$ and $\tilde{g}_s$ are the same as in the previous section.

It is well known that the Riemannian tensor R on a manifold with constant sectional curvature K satisfies the following relation:

$$R(X,Y)Z \equiv \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]} = K(g(Y,Z)X - g(X,Z)Y).$$

We denote such a tensor as R_K. Let us calculate the curvature tensor $R^{h,s}$ [10], corresponding to the connections $\tilde{\nabla}^h$ and $\tilde{\nabla}^s$ on F

$$R^{h,s}(X,Y)\tilde{\xi} \equiv \tilde{\nabla}_X^{h,s}\tilde{\nabla}_Y^{h,s}\tilde{\xi} - \tilde{\nabla}_Y^{h,s}\tilde{\nabla}_X^{h,s}\tilde{\xi} - \tilde{\nabla}_{[X,Y]}^{h,s}\tilde{\xi}$$

where $\tilde{\xi} = \xi + fe$ is a section of F. From equation (11) we obtain

$$\begin{aligned}\tilde{\nabla}_X^s\tilde{\nabla}_Y^s(\xi + fe) = {} & \nabla_X\nabla_Y\xi + \nabla_X(fY) + (Y(f) - g(Y,\xi))X + (X(Y(f) - g(Y,\xi)) \\ & - g(X, \nabla_Y\xi + fY))e\end{aligned}$$

so

$$\begin{aligned}R^s(X,Y)\tilde{\xi} &= \nabla_X\nabla_Y\xi - \nabla_Y\nabla_X\xi + \nabla_X(fY) - \nabla_Y(fX) + Y(f)X - X(f)Y - g(Y,\xi)X \\ &\quad + g(X,\xi)Y + \{(X\circ Y(f) - Y\circ X(f) - X(g(Y,\xi)) + Y(g(X,\xi)) - g(X,\nabla_Y\xi) \\ &\quad + g(Y,\nabla_X\xi)\}e - \nabla_{[X,Y]}\xi - f[X,Y] - ([X,Y](f) - g([X,Y],\xi))e \\ &= R(X,Y)\xi + f(\nabla_XY - \nabla_YX - [X,Y]) - R_1(X,Y)\xi \\ &\quad + (g(\nabla_YX,\xi) - g(\nabla_XY,\xi) + g([X,Y],\xi))e \\ &= R(X,Y)\xi - R_1(X,Y)\xi\end{aligned}$$

due to the equality $K_\nabla(X,Y) = \nabla_XY - \nabla_YX - [X,Y] = 0$ for the torsion K_∇ of the Levi-Civita connection and the condition $\nabla_Xg = 0$. Reasoning in a similar way, we obtain

$$R^h(X,Y)\tilde{\xi} = R(X,Y)\xi - R_{-1}(X,Y)\xi.$$

Thus the connection $\tilde{\nabla}^h$ is flat iff M^n is a space of the constant sectional curvature -1 and the connection $\tilde{\nabla}^s$ is flat iff M^n is a space of the constant sectional curvature 1.

We can verify that the connection $\tilde{\nabla}^h$ conserves the metric $\tilde{g}_h$ and the connection $\tilde{\nabla}^s$ conserves the metric $\tilde{g}_s$. Indeed, let $\tilde{\xi}_i = \xi_i + f_i e$, $i = 1, 2$ be a section of F. Then

$$\tilde{\nabla}^s_X(\tilde{g}_s(\tilde{\xi}_1, \tilde{\xi}_2)) = X(g(\xi_1, \xi_2) + f_1 f_2) = g(\nabla_X \xi_1, \xi_2) + g(\xi_2, \nabla_X \xi_1) + X(f_1 f_2).$$

On the other hand

$$\begin{aligned}\tilde{g}_s\big(\tilde{\nabla}^s_X\tilde{\xi}_1, \tilde{\xi}_2\big) + \tilde{g}_s\big(\tilde{\xi}_1, \tilde{\nabla}^s_X\tilde{\xi}_2\big) &= \tilde{g}_s(\nabla_X\xi_1 + f_1 X + (X(f_1) - g(X, \xi_1))e, \xi_2 + f_2 e)\\ &\quad + \tilde{g}_s(\xi_1 + f_1 e, \nabla_X\xi_2 + f_2 X + (X(f_2) - g(X, \xi_2))e)\\ &= g(\nabla_X\xi_1 + f_1 X, \xi_2) + (X(f_1) - g(X, \xi_1))f_2 + g(\xi_1, \nabla_X\xi_2 + f_2 X)\\ &\quad + (X(f_2) - g(X, \xi_2))f_1\\ &= g(\nabla_X\xi_1, \xi_2) + g(\xi_2, \nabla_X\xi_1) + X(f_1)f_2 + X(f_2)f_1.\end{aligned}$$

The last two equalities give

$$\big(\tilde{\nabla}^s_X\tilde{g}_s\big)(\tilde{\xi}_1, \tilde{\xi}_2) \equiv \tilde{\nabla}^s_X(\tilde{g}_s(\tilde{\xi}_1, \tilde{\xi}_2)) - \tilde{g}_s\big(\tilde{\nabla}^s_X\tilde{\xi}_1, \tilde{\xi}_2\big) - \tilde{g}_s\big(\tilde{\xi}_1, \tilde{\nabla}^s_X\tilde{\xi}_2\big) = 0.$$

A similar reasoning gives $\tilde{\nabla}^h_X\tilde{g}_h = 0$. The conservation of this metric means that the corresponding connection 1-form A is $\mathfrak{so}(n+1)$ valued for $\tilde{\nabla}^s$ and $\mathfrak{so}(n, 1)$ valued for $\tilde{\nabla}^h$ with respect to orthonormal moving frames.

Now let M^n be a simply-connected space with constant sectional curvature ± 1. Considering the standard models for this space as a submanifold of Euclidean (for $K = 1$) or pseudo-Euclidean (for $K = -1$) spaces [10], we see that the bundle F is isomorphic to the trivial bundle $\mathbb{E}^{n,1} \times \mathbb{H}^n$ for $K = -1$ and to $\mathbb{E}^{n+1} \times \mathbb{S}^n$ for $K = 1$, where $\mathbb{E}^{n+1}$ is the $(n+1)$-dimensional Euclidean space and $\mathbb{E}^{n,1}$ is the $(n+1)$-dimensional pseudo-Euclidean space of signature $(n, 1)$. In these cases the connection $\tilde{\nabla}^{h,s}$ is the restriction of the flat Levi-Civita connection for $T\mathbb{E}^{n,1} = \mathbb{E}^{n,1} \times \mathbb{E}^{n,1}$ on to $\mathbb{E}^{n,1} \times \mathbb{H}^n$ or of the flat Levi-Civita connection for $T\mathbb{E}^{n+1} = \mathbb{E}^{n+1} \times \mathbb{E}^{n+1}$ on to $\mathbb{E}^{n+1} \times \mathbb{S}^n$.

4. Discussion

The common point of view [11, 12] is that the Sasaki connection is based only on the internal geometry of surfaces. However, the connection 1-form (3) possesses an additional (with respect to internal geometry) matrix structure. Here we have interpreted this additional structure as the trivial one-dimensional summand in the corresponding vector bundle. In the case of the constant sectional curvature, this summand becomes a normal bundle of the hypersurfaces. Thus our interpretation means a 'virtual' imbedding of the initial space M^n into the space $\mathbb{E}^{n+1}$ or $\mathbb{E}^{n,1}$. This 'virtual' imbedding becomes actual when M^n is a space with constant sectional curvature ± 1.

In [13, 14], a multi-dimensional generalization of the sine-Gordon equation $u_{xy} = \sin u$ was given as an imbedding condition of M^n into $\mathbb{E}^{2n-1}$. On the other hand, it is well known [1] that the condition $dA + A \wedge A$ for the matrix 1-form A given by (3) is equivalent to the sine-Gordon equation whenever differential forms ω^1, ω^2 are parametrized by the function u in a definite way. The generalization of the Sasaki connection given in this paper seems to be quite natural, so it can lead to another multi-dimensional generalization of the sine-Gordon equation.

After completing this paper, the author has received a letter from Jack Lee of University of Washington (to whom the author expresses his deep gratitude), who has pointed to the paper of Min-Oo [15]. In that paper, the connection $\tilde{\nabla}^h$ on $TM \oplus E$ was constructed under the name *hyperbolic Cartan connection* in order to prove the hyperbolic version of the positive mass theorem. There are no links with the theory of integrable partial differential equations and particularly with the Sasaki construction. Thus, the present paper establishes the connection between the pure geometrical construction of Min-Oo and the well-known construction from the theory of integrable partial differential equations.

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